Closing today: HW_6A,6B,6C (7.5, 7.7)
Closing next Wed: HW_7A,7B,7C, (7.8, 8.1)
Exam 2 is Thurs, May 18th (6.4,6.5,7.1-7.8,8.1)
Office Hours Today: 1:30-3:00pm Smith 309.

Entry Task: Use n = 3 and the trapezoid rule to approximate:

$$\int_0^3 \sqrt{100 - x^3}\,dx$$

7.7 Approximating Integrals:

Despite our best efforts in 7.1-7.5**,** <u>the vast majority of integrals CANNOT be done with any of our methods</u>.

To approximate $\int_a^b f(x)dx$

1. Pick **n = number of subdivisions**.
   Compute $\Delta x = \frac{b-a}{n}$.
2. Label the tick marks: $x_i = a + i\Delta x$
3. Use one of these:

$$L_n = \Delta x[f(x_0) + f(x_1) + \cdots + f(x_{n-1})] \quad \text{(Left endpoint)}$$
$$R_n = \Delta x[f(x_1) + f(x_2) + \cdots + f(x_n)] \quad \text{(Right endpoint)}$$
$$M_n = \Delta x[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)] \quad \text{(Midpoint)}$$

New - Trapezoid Rule: (all the “middle terms” are multiplied by 2)

$$T_n = \frac{1}{2}\Delta x[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

New - Simpson’s Rule: *n* must be even! (Alternating multiplying middle terms by 4 and 2)

$$S_n = \frac{1}{3}\Delta x[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Example: (Note: None of our methods can integrate this)

Estimate $\int_0^3 \sqrt{100 - x^3}dx$

$$L_3 = (1)\left[\sqrt{100-(0)^3} + \sqrt{100-(1)^3} + \sqrt{100-(2)^3}\right] \approx 29.5415$$

$$R_3 = (1)\left[\sqrt{100-(1)^3} + \sqrt{100-(2)^3} + \sqrt{100-(3)^3}\right] \approx 28.0855$$

$$M_3 = (1)\left[\sqrt{100-(0.5)^3} + \sqrt{100-(1.5)^3} + \sqrt{100-(2.5)^3}\right] \approx 29.0091$$

NEW – Trapezoid rule with n = 3.

$$T_3 = \frac{1}{2}(1)\left[\sqrt{100-(0)^3} + 2\sqrt{100-(1)^3} + 2\sqrt{100-(2)^3} + \sqrt{100-(3)^3}\right] \approx 28.8135$$

NEW – Simpson's rule with n = 6 (n must be even)

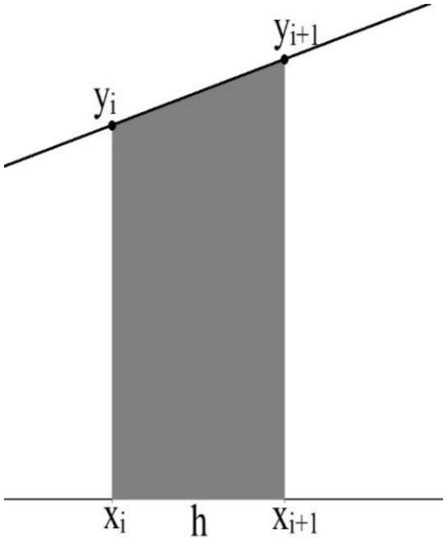
$$S_6 = \frac{1}{3}\cdot\frac{1}{2}\Big[\sqrt{100-(0)^3} + 4\sqrt{100-(0.5)^3} + 2\sqrt{100-(1)^3} + 4\sqrt{100-(1.5)^3}$$
$$+2\sqrt{100-(2)^3} + 4\sqrt{100-(2.5)^3} + \sqrt{100-(3)^3}\Big] \approx 28.9441$$

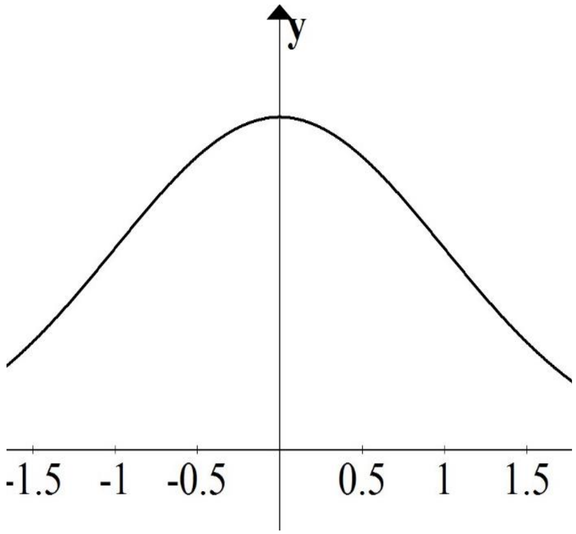
"Actual" Value (to 8 places after the decimal): 28.94418784

7.7 Quick Derivation Notes

Trapezoid Rule:

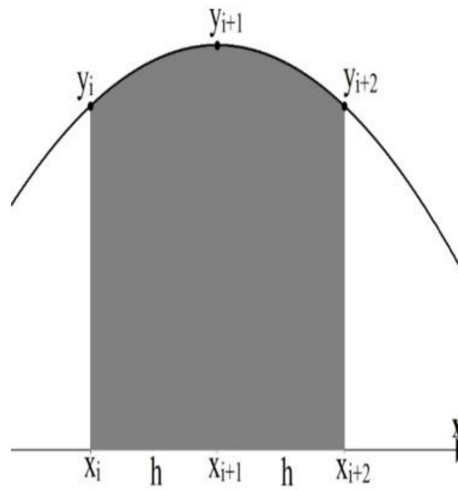
$$\text{Shaded Area } = \frac{h}{2}(y_i + y_{i+1})$$

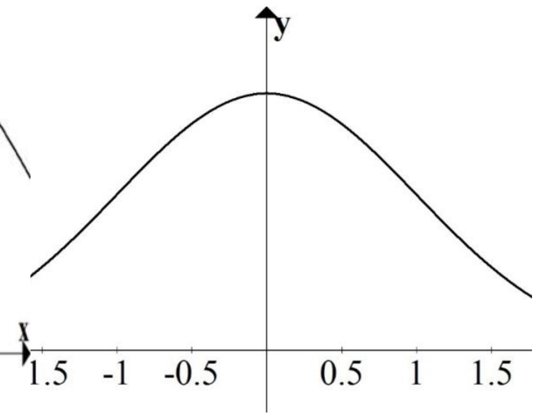




Simpson's Rule: If the curve below is a ***parabola*** (y = ax^2 + bx + c) that goes through the three indicated points, then

$$\text{Shaded Area} = \frac{h}{3}(y_i + 4y_{i+1} + y_{i+2})$$





Example: With **n = 4**, use both new methods to approximate (just set up)

$$\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-\frac{1}{2}x^2}\,dx$$

$$\frac{1}{2}\Delta x[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)]$$

$$=$$

$$\frac{1}{3}\Delta x[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)]$$

$$=$$

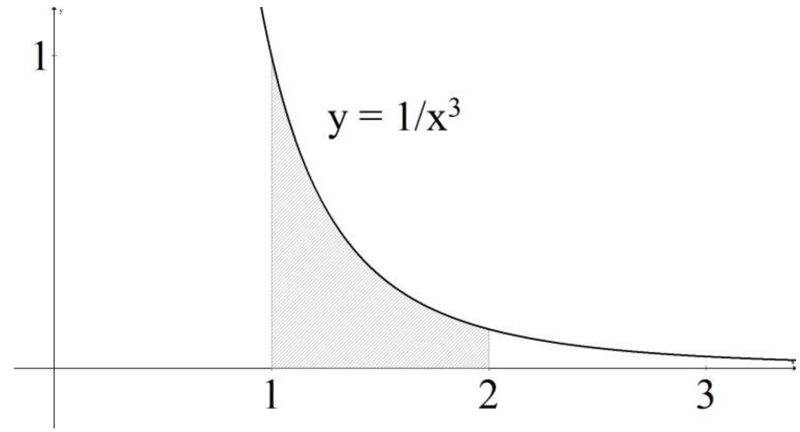
7.8 Improper Integrals

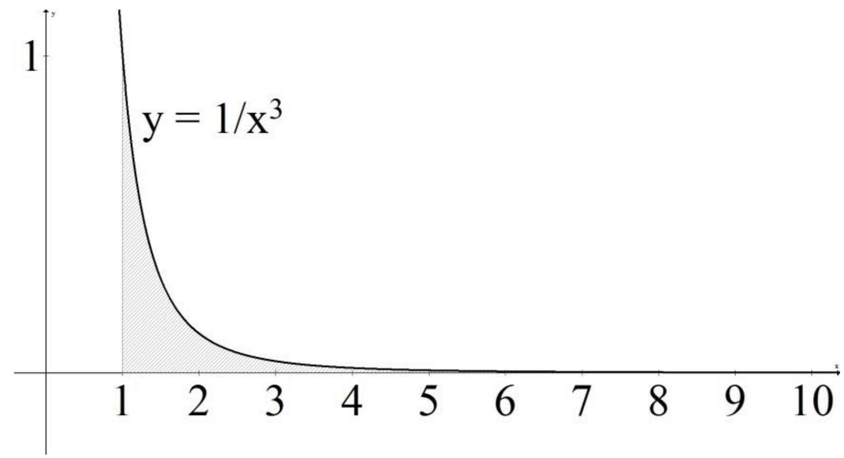
Motivation:

Consider the function $f(x) = \frac{1}{x^3}$.

Come the area under this function from…

1. x = 1 to x = t
2. x = 1 to x = 2
3. x = 1 to x = 100





Def'n: *Improper type 1 - infinite integral of integration*

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$$

If the limit exists and is finite, then we say the integral *converges*. Otherwise, we say it *diverges*.

Example:

1. $\displaystyle\int_0^\infty \frac{1}{x^3}\,dx =$

Example:

2. $\int_{-1}^{\infty} e^{-2x}\,dx =$

3. $\int_{1}^{\infty} \frac{1}{x}\,dx =$

Def'n:

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{r\to-\infty} \int_{r}^{0} f(x)dx + \lim_{t\to\infty} \int_{0}^{t} f(x)dx$$

In this case, we say it *converges* only if both limits separately exist and are finite.

3. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx$

Def'n: *Improper type 2 - infinite discontinuity*

If f(x) has a discontinuity at x = a, then

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

If f(x) has a discontinuity at x = b, then

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

If the limit exists and is finite, then we say the integral *converges*. Otherwise, we say it *diverges*.

Example:

1. $\int_0^1 \frac{1}{\sqrt{x}} dx =$

Example:

$$2.\ \int_0^2 \frac{x}{x-2}\,dx =$$

If f(x) has a discontinuity at x = c which is **between** a and b, then

$$\int_a^b f(x)dx = \lim_{r\to c^-}\int_a^r f(x)dx + \lim_{t\to c^+}\int_t^b f(x)dx$$

In this case, we say it *converges* only if both limits separately exist and are finite.

3. $\displaystyle\int_0^{\pi} \frac{1}{\cos^2(x)}\,dx =$

Limits Refresher

1. If stuck, plug in values "near" t.
2. Know your basic functions/values:

$$\lim_{t\to\infty} \frac{1}{t^a} = 0, \qquad \text{if } a > 0.$$
$$\lim_{t\to\infty} \frac{1}{e^{at}} = 0, \qquad \text{if } a > 0.$$
$$\lim_{t\to\infty} t^a = \infty, \qquad \text{if } a > 0.$$
$$\lim_{t\to\infty} \ln(t) = \infty.$$
$$\lim_{t\to 0^+} \ln(t) = -\infty.$$

3. For indeterminant forms, use algebra and/or L'Hopital's rule

Examples:

$$\lim_{t\to 1} \frac{t^2+2t-3}{t-1} =$$

$$\lim_{t\to\infty} \frac{\ln(t)}{t} =$$

$$\lim_{t\to\infty} t^2 e^{-3t} =$$

Aside:

A few general notes on **comparison**:

Suppose you have two functions f(x) and g(x) such that $0 \le g(x) \le f(x)$ for all values.

(a) If $\int_1^\infty f(x)dx$ converges,
 then $\int_1^\infty g(x)dx$ converges.

(b) If $\int_1^\infty g(x)dx$ diverges,
 then $\int_1^\infty f(x)dx$ diverges.

You can verify that

$$\int_1^\infty \frac{1}{x^p}dx\,, \qquad \text{converges for } p > 1.$$

$$\int_1^\infty e^{px}\,dx\,, \qquad \text{converges for } p < 0.$$

And you can compare off of these to sometimes quickly tell is something is converging or diverging (without calculating anything)